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LIPMAN BERS

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The purpose of this note is to strengthen the results of [3] and to indicate a very brief derivation of some theorems announced without proof in [1] and [3].

We begin by indicating a correction to [3]. Let S_1 and S_2 be Riemann surfaces, f an orientation preserving (orientation reversing) homeomorphism of bounded excentricity of S_1 onto S_2 and $[f]$ the homotopy class of f; then $(S_1,[f],S_2)$ is called an even (odd) coupled pair of Riemann surfaces. The definition of equivalence of such pairs given in [3] is imprecise and garbled by misprints. The correct definition reads: $(S_1,[f],S_2)$ and $(S_1',[f'],S_2')$ are called <u>equivalent</u> if there exist conformal homeomorphisms h_1 and h_2 with $h_1(S_1) = S_1', h_2(S_2) = S_2'$ and $[h_2f] = [f'h_1]$; the two pairs are called <u>strongly equivalent</u> if $S_2' = S_2$ and there exists a conformal homeomorphism h with $h(S_1) = S_1'$ and $[f] = [h'f]$. If S_o is a Riemann surface, then the Teichmüller space $T(S_o)$ can be thought of as the set of strong equivalence classes of even pairs $(S,[f],S_o)$ (and not of simple equivalence classes as stated in [3]).[1]

From now on we assume that S_o is a fixed closed Riemann surface of genus $g > 1$, and we write T instead of $T(S_o)$. T has a natural complex analytic structure and can be represented as a bounded domain, homeomorphic to a ball, in complex number space with coordinates (moduli) $\tau_1, \ldots, \tau_{3g-3}$ (cf.[1,2]). Points of T will be denoted by τ. We may assume that S_o is given as the unit disc modulo a fixed-point-free Fuchsian group, and that $\tau=0$ corresponds to the pair $(S_o,[\text{identity}], S_o)$.

THEOREM I. <u>One can associate to every</u> $\tau \in T$ <u>a bounded Jordan domain</u> $D(\tau)$ <u>and</u> 2g <u>Möbius transformations</u> $z \to A_j(z,\tau)$, $z \to B_j(z,\tau)$, $j = 1, \ldots, g$, <u>such that the following conditions are satisfied</u>.

(i) <u>The boundary curve of</u> $D(\tau)$ <u>admits the parametric representation</u> $z = \sigma(e^{i\theta}, \tau)$, $0 \leq \theta \leq 2\pi$, σ <u>depending holomorphically on</u> τ. $D(0)$ <u>is the unit disc</u>.

(ii) <u>The</u> A_j <u>and</u> B_j <u>depend holomorphically on</u> τ <u>and satisfy the relation</u>

(1) $$\prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} = 1$$

<u>For every fixed</u> $\tau \in T$ <u>they generate, with the single defining relation</u> (1), <u>a fixed-point-free discrete group</u> $G(\tau)$ <u>of conformal self-mappings of</u> $D(\tau)$, <u>so that</u> $S(\tau) = D(\tau)/G(\tau)$ <u>is a closed Riemann surface of genus</u> g. $S(0)$ <u>is the surface</u> S_0.

(iii) Denote by $\alpha(\tau)$ the basis of the fundamental group of $S(\tau)$ defined by $A_1, \ldots, B_g$, and by f_τ a quasiconformal mapping of $S(\tau)$ onto $S(0)$ which takes $\alpha(\tau)$ into $\alpha(0)$. Then the point τ corresponds to the pair $(S(\tau), [f_\tau], S_0)$.

This statement differs from Theorem 2 in [3] primarily by the boundedness condition for $D(\tau)$ and can be obtained from that theorem without much difficulty.

We denote by M the domain in complex number space of 3g-2 dimensions which consists of points (z,τ) with $z \in D(\tau)$ and $\tau \in T$. By Theorem 3 in [3] M is holomorphically equivalent to $T(S_0 - \{p\})$ for a fixed $p \in S_0$.

We denote by $W_q(\tau)$ the (complex) vector space of holomorphic functions $\varphi(z)$, $z \in D(\tau)$, for which $\varphi(z)dz^q$ is invariant under $G(\tau)$; this is the same as the space of q-dimensional holomorphic differentials on $S(\tau)$, so that $\dim W_q(\tau) = 0, 1, g$, or $(2q-1)(g-1)$ according to whether $g \leq 0$, $g=0, g=1$, or $g>1$. In $W_1(\tau)$ there exist g distinguished elements, $p_k(z,\tau)$, determined by the conditions

$$\int_{z}^{A_i(z,\tau)} p_k(z';\tau)\,dz' = \delta_{ik}; \tag{2}$$

these correspond to the normalized Abelian differentials of the first kind on $S(\tau)$ belonging to the 'canonical' homology basis $a(\tau)$ determined by $\alpha(\tau)$. The period matrix of $S(\tau)$ belonging to $a(\tau)$ will be denoted by $Z(\tau)$. It has the elements

$$Z_{ik}(\tau) = \int_{z}^{B_i(z,\tau)} p_k(z',\tau)\,dz'$$

and is a point in the Siegel space of symmetric matrices with positive definite imaginary part.

We denote by $\underline{W}_q$ the vector space of holomorphic functions $\Phi(z,\tau)$, $(z,\tau) \in M$, which belong to $W_q(\tau)$ for every fixed $\tau \in T$.

THEOREM II. <u>Every element of $W_q(\tau)$ is a restriction of an element of $\underline{W}_q$.</u>

Proof. Assume that $q > 2$. Let C_j, $j=1, 2, \ldots$, be a complete system of non equivalent (with respect to (1)) words in the letters $A_1, \ldots, B_g$. If $P(t)$ is a polynomial, then the Poincaré series

$$\sum_{j=1}^{\infty} P(C_j(z,\tau))(\partial C_j(z,\tau)/\partial z)^q \tag{3}$$

converges normally in M and its sum belongs to $\underline{W}_q$. On the other hand, since $D(\tau)$ is a bounded Jordan domain and $G(\tau)$ has a compact fundamental region, Theorem 4 in [4] implies that, for a fixed τ, every element of $W_q(\tau)$ is of the form (3).

For $q = 1$ we shall show that every p_j belongs to $\underline{W}_1$ (i.e. that the normalized Abelian differentials are holomorphic functions of the moduli).

THEOREM III. <u>The functions</u> $p_k(z,\tau)$, $k=1, \ldots, g$, <u>are holomorphic in</u> M.

Proof. It suffices to consider p_1. We shall show that in a neighborhood of a fixed but arbitrary point $\tau_0 \in T$ we have an identity of the form

$$(4)\qquad p_1(z,\tau) = \phi_1(z,\tau)^{-1} \sum_{j=1}^{5g-5} c_j(\tau)\, \phi_{1j}(z,\tau)$$

where the c_j are holomorphic, $\phi_1 \in \underline{W}_2$, and the ϕ_{1j} are elements of $\underline{W}_3$. We first choose ϕ_1 so that $\phi_1(z,\tau_0)$ vanishes at $4g-4$ points z_i which are not equivalent under $G(\tau_0)$. This is possible since the 'general' holomorphic quadratic differential on $S(\tau)$ has only simple zeros (Bertini) and hence exactly $4g-4$ of those. There exist $4g-4$ holomorphic functions $\zeta_i(\tau)$ defined near τ_0, such that $\zeta_i(\tau_0)=0$ and $\phi_1(z_i + \zeta_i(\tau),\tau) = 0$. In order that the right hand side of (4) belong to $W_1(\tau)$ it is necessary and sufficient that

$$\sum_{j=1}^{5g-5} c_j(\tau)\, \phi_{1j}(z_i + \zeta_i(\tau),\tau) = 0,\quad i = 1, \ldots, 4g-4,$$

and one sees at once that any $4g-5$ of these equations imply the $(4g-4)$-th. In order that (4) hold near τ_0 the c_j must satisfy g

additional linear equations which are obtained from (1) by setting $k=1$ and choosing a fixed point z and fixed paths of integration, avoiding the points z_i. The resulting linear system, with holomorphic coefficients, for the unknown functions c_j, is uniquely solvable at τ_o if the functions $\overset{\tau}{\theta}_1, \ldots, \overset{\tau}{\theta}_{15g-5}$ are chosen so as to be linearly independent for $\tau = \tau_o$. In this case the equations are uniquely solvable for τ close to τ_o, and the solutions depend holomorphically on τ.

We proceed to derive some consequences from Theorems II and III.

(a) The functions

$$f_{ij} = p_i/p_j, \quad f_{ijk} = f_k^{-1}\,\partial \log f_{ij}/\partial z$$

are meromorphic in M. This proves Theorem J in [1]. It is classical that every meromorphic function in $D(\tau)$ which is automorphic under $G(\tau)$ can be expressed rationally in terms of the functions f_{ij}, f_{ijk} (and even in terms of the f_{ij} alone if $S(\tau)$ is not hyperelliptic). Thus we obtain a proof of Theorem 4 in [3] which asserts the existence of finitely many meromorphic functions of the moduli and of an additional complex variable which uniformize simultaneously all algebraic curves of genus $g > 1$.

(b) Let us choose $(2q-1)(g-1)$ elements of $\underline{W}_q$, $q>1$ (or g elements of $\underline{W}_1$) which are linearly independent for $\tau = \tau_o$, and let $w(z,\tau)$ denote their Wronskian with respect to z. For a fixed τ close to τ_o the zeros of $w(z,\tau)$ are precisely the Weierstrass points of $S(\tau)$, in the classical sense if $q = 1$, in the sense of Petersson if $q > 1$ (cf. the definition in [4]). Since w is a holomorphic function in M

we conclude that *the Weierstrass points on a closed Riemann surface depend holomorphically on the moduli* (cf. Rauch [6], Röhrl [7]).

(c) Now let $w(z,\tau)$ denote the Wronskian of an arbitrary set of $\dim W_q(\tau)$ elements of $\underline{W}_q$ and let N denote the set of those $\tau \varepsilon T$ for which $w(z,\tau) \equiv 0$. If z_o is not a Weierstrass point of $S(\tau_o)$, then there is a neighborhood of τ_o in which the points of N are precisely the zeros of $w(z_o,\tau)$. We conclude that N *is either empty, or the whole domain* T, *or an analytic subvariety of* T *of codimension* 1.

(d) Let H denote the set of those $\tau \varepsilon T$ for which $S(\tau)$ is hyperelliptic. For $\tau \varepsilon T - H$ every element of $W_q(\tau)$ can be written as a homogeneous polynomial in the p_j (M. Noether). For $\tau \varepsilon H$ the subspace of $W_q(\tau)$ consisting of homogeneous polynomials in elements of $W_1(\tau)$ has dimension $q(g-1)-1$. But H is an analytic subvariety of T of dimension $2g-1$, so that, noting (c), we obtain the following complement to Noether's theorem: *for* $g > 2$ *and* $q > 1$ *there exist no fixed set of* $(2q-1)(g-1)$ *homogeneous polynomials of degree* q *in normalized Abelian differentials of the first kind which spans the space of holomorphic differentials of dimension* q *on all non-hyperelliptic closed Riemann surfaces of genus* g.

(e) *The mapping* $\tau \to Z(\tau)$ of the Teichmüller space into the Siegel space *is holomorphic*. This follows at once from Theorem III, and also by using the coordinates in T defined in [1] in conjunction with Rauch's variational formulas [5]. These formulas also show

that the mapping of T into a (3g-3) -dimensional subspace of the Siegel space,

$$\tau \to \left\{ \sum_{i,k=1}^{3g-3} \gamma_{j,ik} Z_{ik}(\tau), \quad j = 1, \ldots, 3g-3 \right\}$$

is one-to-one near a point τ_o if and only if the 3g-3 functions

$$\sum_{i,k=1}^{3g-3} \gamma_{j,ik} p_i(z,\tau_o) p_k(z,\tau_o)$$

are linearly independent. This shows that near every non-hyperelliptic surface a properly chosen set of 3g-3 periods Z_{ik} can serve as a set of local moduli (Rauch). On the other hand, (d) implies a complementto Rauch's theorem: <u>no fixed set of 3g-3 linear combinations of periods can serve as a set of moduli near every non-hyperelliptic closed Riemann surface of genus</u> $g > 2$.

FOOTNOTE

1) We also note the following errata to [2,3]. On p. 94, 1.19, replace (ζ) by $\mu(\zeta)$. On p. 96, 1.15, replace the subscript j by 2j. On p. 97, 1.21, replace C_r by C^r. On p. 100, 1.4, replace 'covering' by 'covering space'. On p. 103, equation (9) replace the exponent 3g-3n+n by 3g-3+n.

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New York University